Second Order Linear Equation

Missing *x* or *y*, Reduction of Order:

Solve the following equations:

1) $x^2y''+xy'=\frac{1}{x}\ \left(x\neq 0\right)$

2) $y''\tan x-1=y'\ ,\ \left(\cos x\neq 0\right)$

3) $2xy'y''-\left(y'\right)^2+1=0$

4) $y''x\ln x=y'$

5) $xy''=x^2e^x+y'$

6) $y\cdot y''+(y')^2=0$

7) $2y''y-\left(y'\right)^2=1$

8) $y''\tan y=2\left(y'\right)^2\ \left(\cos y\neq 0\right)$

Final Answers:

1) $y=\frac{1}{x}+C_1\cdot\ln x+C_2$

2) $y=-x+C_1\cdot\cos x+C_2$

3) $y=\pm x+C_3$

4) $y=C_3$

5) $y=e^x\left(x-1\right)+C_1\frac{x^2}{2}+C_2$

6) $\frac{y^2}{2}=cx+k\ ,\ y=c$

7) $y=\frac{1}{c}\left[\frac{c^2\left(x+k\right)^2}{4}+1\right]$

8) $\cot y=-\left(cx+k\right)\ ,\ y=c$

Linear, Homogeneous, Constant Coefficients:

Solve the following equations:

1) $y''-100y=0$

2) $y''-4y'=0$

3) $y''-8y'+7y=0$

4) $4z''+z'-5z=0$; $z(0)=1$, $z'(0)=1$

5) $y''-2y'+y=0$

6) $4\frac{d^2x}{dt^2}+4\frac{dx}{dt}+x=0$

7) $y''+10y'+125y=0$

8) $y''+4y=0$

9) $y''-2y'+10y=0$; $y(0)=0,\ y'(0)=3$

Final Answers:

1) $y=c_1e^{10x}+c_2e^{-10x}$

2) $y=c_1+c_2e^{4x}$

3) $y=c_1e^{x}+c_2e^{7x}$

4) $z=e^{x}$

5) $y=c_1e^{x}+c_2xe^{x}$

6) $x=c_1e^{-0.5t}+c_2te^{-0.5t}$

7) $y=e^{-5x}\left[c_1\cos 10x+c_2\sin 10x\right]$

8) $y=c_1\cos 2x+c_2\sin 2x$

9) $y=e^{x}\sin 3x$

Linear, Nonhomogeneous, Constant Coefficients - Method of Undetermined Coefficients:

Solve the following equations:

1) $y''+5y'+6y=22x+6x^2$

2) $y''-2y'+y=e^{2x}$; $y(0)=2, y'(0)=7$

3) $y''-y'-2y=4\sin 2x$

4) $y''-2y=xe^{-x}$

5) $y''-y=3e^{2x}\cos x$

6) $z''+z=\sin x$

7) $y''-3y'+2y=2x^2+e^x+2xe^x+4e^{3x}$

8) $y''+3y'=9x$

9) $y''-3y'+2y=e^x$

10) $y''-2y'=6x^2-2x$

11) $x''+5x'+6x=e^{-t}+e^{-2t}$

12) $y''+2y'+5y=e^{-x}\sin 2x$

Final Answers:

1) $y=c_1e^{-3x}+c_2e^{-2x}+x^2+2x-2$

2) $y=e^x+4xe^x+e^{2x}$

3) $y=c_1e^{2x}+c_2e^{-x}+\frac{1}{5}\sin 2x-\frac{3}{5}\cos 2x$

4) $y=c_1e^{-\sqrt{2}x}+c_2e^{\sqrt{2}x}+(2-x)e^{-x}$

5) $y=c_1e^{-x}+c_2e^{x}+\frac{3}{10}e^{2x}\cos x+\frac{3}{5}e^{2x}\sin x$

6) $z=c_1\cos x+c_2\sin x-\frac{1}{2}x\cos x$

7) $y=c_1e^{x}+c_2e^{2x}+x^2+3x+3.5-x^2e^x-3xe^x+2e^{3x}$

8) $y=c_1+c_2e^{-3x}+\frac{3}{2}x^2-x$

9) $y=c_1e^{x}+c_2e^{2x}-xe^x$

10) $y=c_1e^{-3x}+c_2e^{-2x}-x^2-x-x^3$

11) $x=c_1e^{-2t}+c_2e^{-3t}+\frac{1}{2}\cdot e^{-t}+te^{-2t}$

12) $y=e^{-x}\left[c_1\cos 2x+c_2\sin 2x\right]-{}^{1}\!/_{4}\,x\cdot e^{-x}\cos 2x$

Linear, Nonhomogeneous, Constant Coefficients - Method of Variation of Parameters:

Solve the following equations:

1) $y''+y=\dfrac{1}{\sin x}$

2) $y''+4y'+4y=e^{-2x}\ln x$

3) $y''+2y'+y=3e^{-x}\sqrt{x+1}$

4) $y''-2y'+y=\dfrac{e^{x}}{x}$; $y(1)=0$, $y'(1)=0$

5) $y''-3y'+2y=\dfrac{1}{1+e^{-x}}$

6) $y''+4y=\sec 2x$

Final Answers:

1) $y=c_1\cos x+c_2\sin x-\cos x\cdot x+\sin x\cdot\ln|\sin x|$

2) $y=c_1e^{-2x}+c_2xe^{-2x}-e^{-2x}\dfrac{x^2}{2}\left[\ln x-\dfrac{1}{2}\right]+x^2e^{-2x}\left[\ln x-1\right]$

3) $y=c_1e^{-x}+c_2xe^{-x}-e^{-x}\left[\dfrac{6\left(\sqrt{x+1}\right)^5}{5}-\dfrac{6\left(\sqrt{x+1}\right)^3}{3}\right]+xe^{-x}\left[2\left(x+1\right)^{3/2}\right]$

4) $y=e^{x}-xe^{x}+xe^{x}\ln x$, $\left(x>0\right)$

5) $y=c_1e^{x}+c_2e^{2x}+e^{x}\ln\left(1+e^{-x}\right)+e^{2x}\left[\ln(1+e^{-x})-(1+e^{-x})\right]$

6) $y=c_1\cos 2x+c_2\sin 2x+\dfrac{1}{4}\cos 2x\ln|\cos 2x|+\dfrac{1}{2}\sin 2x\cdot x$

Linear, Homogeneous, Non-Constant Coefficients - 2nd Solution Method:

Solve the following equations:

1) $y''+\tan x\cdot y'-\left(2\tan x+4\right)y=0$

2) $\left(1-x^2\right)y''+2xy'-2y=0$, $\left(x>0\right)$

3) $\left(1+x^2\right)y''-1.5xy'+y=0$, $\left(x>2\right)$

Final Answers:

1) $y=c_1e^{2x}+c_2e^{-2x}\left(\sin x-4\cos x\right)$

2) $y=c_1x+c_2\left(x^2+1\right)$

3) $y=c_1\left(x^2-2\right)+c_2\left(x^2-2\right)h\left(x\right)$ where $h\left(x\right)$ is defined in the movie.

The Wronskian and Its Uses:

Solve the following equations:

1) Is it possible that $y_1(x) = e^x,\ y_2(x) = \sin x$ are two solutions of a linear homogeneous 2nd order ODE $y'' + p(x)y' + q(x)y = 0$ with continuous coefficients $p(x), q(x)$ on the interval $[0, \pi]$?

2) Answer the following questions:

a. Show that the functions $y_1(x) = \sin x^2,\ y_2(x) = \cos x^2$ are l.i. solutions of the equation $xy'' - y' + 4x^3 y = 0$ on any interval.

b. Show that the Wronskian is zero only for $x = 0$.

c. Joe claims that we have a contradiction to one of our propositions.

  i. Which proposition is Joe referring to?

  ii. Is Joe right?

3) It is easily verified that the functions $y_1(x) = xe^x\ \ y_2(x) = e^{-x}$ are solutions of the equation $y'' - \frac{2}{1+2x}y' - \frac{2x+3}{1+2x}y = 0$ on the interval $\left(-\frac{1}{2}, \infty\right)$.
Are these functions l.i. on the interval?

4) We're given two functions $y_1 = x^3,\ y_2 = \left|x^3\right|$ on the interval $I = [-4, 4]$.

a. Compute the Wronskian of the functions on I.

b. Are the functions 1,2 on I ?

c. Could the functions be two solutions of an ODE $y'' + p(x)y' + q(x)y = 0$ with continuous coefficients?

d. Note that our functions are solutions of the ODE $xy'' - 2y' = 0$ on I.
Does that contradict the result of part c?

5) Answer the following questions:

a. Let $y_1(x)$, $y_2(x)$ be functions which are twice continuously differentiable on an interval *I* and such that their Wronskian is nonzero on *I*.
Prove that there exists an ODE $y''+p(x)y'+q(x)y=0$ with continuous coefficients on *I* such that $y_1(x)$, $y_2(x)$ are two of its solutions.

b. Find an equation $y''+p(x)y'+q(x)y=0$ with continuous coefficients on $x>0$ such that $y_1(x)=x^2$, $y_2(x)=x^4$ are two of its solutions.

Final Answers:

1) $\cos x = \sin x$

2) a. $W(x) = -2x$ b. This was already shown at a. c. prove.

3) Prove.

4) a. $W=0$ always. b. prove. c. no d. no contradiction

5) a. prove. b. $y''-\frac{5}{x}y'+\frac{8}{x^2}y=0$, $(x>0)$

Sturm-Liouville Problems:

Solve the following equations:

1) $\begin{cases} y''+\lambda y=0 \ , \ 0\le x\le 1 \\ y'(0)=0 \\ y'(1)=0 \end{cases}$

2) $\begin{cases} y''+\lambda y=0 \ , \ 0\le x\le 1 \\ y(0)=0 \\ y(1)+y'(1)=0 \end{cases}$

3) $\begin{cases} y''+\lambda y=0 \ , \ 0\le x\le 1 \\ y(0)+y'(0)=0 \\ y(1)=0 \end{cases}$

4) $\begin{cases} y''+\lambda y=0 \ , \ 0\le x\le l \\ y(0)=0 \\ y'(l)=0 \end{cases}$

5) $\begin{cases} y''+\lambda y=0 \ , \ 0\le x\le \pi \\ y'(0)=0 \\ y(\pi)=0 \end{cases}$

6) $\begin{cases} y''-2y'+(1+\lambda)y=0 \ , \ 0\le x\le 1 \\ y(0)=0 \\ y(1)=0 \end{cases}$

Final Answers:

1) $y=0$

2) $y=0$

3) $y=0$

4) $y=0$

5) $y=0$

6) $y=0$