

Solving Linear ODEs with Power Series

Nonhomogeneous Equation around Regular Point:

Solve the following equations:

1) $y'' - 2x^2 y' + 4xy = x^2 + 2x + 2$; $y(0) = 3$, $y'(0) = 12$

2) $y'' - xy = 0$; $y(0) = 1$, $y'(0) = 2$

3) $(1 - x^2)y'' - 2xy' + 2y = 0$

4) $(x^2 + 4)y'' + xy = x + 2$

5) $y'' + (x - 1)y' + (2x - 3)y = 0$

6) $y'' + ty = e^{t+1}$; $y(0) = a_0 = 1$, $y'(0) = a_1 = 2$

7) $y'' + (t - 1)y' + (2t - 3)y = 0 \rightarrow y'' + ty' - y' + 2ty - 3y = 0$

8) $y''(x) + (x - 1)y(x) = e^x$; $y(1) = 1$, $y'(1) = 2$

by developing the solution as a power series around $x = 1$.

9) $y'' + xy' + (2x - 1)y = 0$; $y(-1) = 2$, $y'(-1) = -2$

Final Answers:

$$1) a_n = \frac{2n-10}{(n-1)n} a_{n-3}, \quad (n \geq 5)$$

$$2) a_n = \frac{1}{(n-1)n} a_{n-3}, \quad (n \geq 3)$$

$$3) a_n = \frac{n-3}{n-1} a_n, \quad (n \geq 2)$$

$$4) a_n = \frac{-1}{4(n-1)n} a_{n-3} - \frac{(n-2)(n-3)}{4(n-1)n} a_{n-2}, \quad (n \geq 4)$$

$$5) a_n = \frac{1}{n} a_{n-1} - \frac{n-5}{n(n-1)} a_{n-2} - \frac{2}{n(n-1)} a_{n-3}, \quad (n \geq 3)$$

$$6) a_n = \frac{e}{n(n-1)(n-2)!} - \frac{a_{n-3}}{n(n-1)}, \quad (n \geq 3)$$

$$7) a_n = \frac{1}{n} a_{n-1} - \frac{n-5}{n(n-1)} a_{n-2} - \frac{2}{n(n-1)} a_{n-3}, \quad n \geq 3$$

$$8) a_n = \frac{e - a_{n-3}(n-2)!}{n!}, \quad n \geq 3$$

$$9) a_n = \frac{1}{n} a_{n-1} - \frac{n-5}{n(n-1)} a_{n-2} - \frac{2}{n(n-1)} a_{n-3}, \quad n \geq 3$$

Homogeneous Equation around Regular-Singular Point:

Solve the following equations:

1) $3x^2y'' + 2xy' + x^2y = 0$

2) $2x^2y'' + 7x(x+1)y' - 3y = 0$

3) $2x^2y'' - xy' + (x-5)y = 0$

4) $3x^2y'' - xy' + y = 0$

5) $x^2y'' + xy' + x^2y = 0$

6) $x^2y'' - xy' + y = 0$

7) $x^2y'' + x(x+2)y' - 2y = 0$

8) $x^2y'' + x(x-2)y' + 2y = 0$

Final Answers:

1) $y = k_1x^{1/3} \left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 + \dots \right) + k_2 \left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 + \dots \right)$

2) $y = k_1x^{1/2} \left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \dots \right) + k_2x^{-3} \left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 \right)$

3) $y = k_1x^{-1} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \dots \right) + k_2x^{2.5} \left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \dots \right)$

4) $y = k_1x + k_2x^{1/3}$

5) $y = c_1y_1 + c_2y_2$

6) $y(x) = c_1x + c_2x \ln x$

7) $y(x) = \frac{c_1}{x^2} \left(1 - x + \frac{1}{2}x^2 - e^{-x} \right) + c_2x^{-2}e^{-x}$

8) $y(x) = c_1y_1 + c_2y_2$