Systems of Linear ODEs

Linear Algebra - Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of the matrix:

1) $A=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$

2) $A=\begin{pmatrix}-1&3&0\\3&-1&0\\-2&-2&6\end{pmatrix}$

3) $A=\begin{pmatrix}4&-1&-1\\1&2&-1\\1&-1&2\end{pmatrix}$

4) $A=\begin{pmatrix}1&-1&4\\3&2&-1\\2&1&-1\end{pmatrix}$

5) $A=\begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}$

6) $A=\begin{pmatrix}1&1\\4&1\end{pmatrix}$

7) $A=\begin{pmatrix}3&-2\\4&-1\end{pmatrix}$

8) $A=\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix}$

Final Answers:

1) $x=0$, $x=1$, $x=2$,

$v_{x=0}=(-1,0,1)$, $v_{x=1}=(0,1,0)$, $v_{x=2}=(1,0,1)$

2) $x=6$, $x=2$, $x=-4$

$v_{x=6}=(0,0,1)$, $v_{x=2}=(1,1,1)$, $v_{x=-4}=(-1,1,0)$

3) $x_1=2$, $x_2=3$

$v_{x=2}=(1,1,1)$, $v_{x=3}^{(1)}=(1,0,1)$, $v_{x=3}^{(2)}=(1,1,0)$

4) $x=1$, $x=3$, $x=-2$

$v_{x=1}=(-1,4,1)$, $v_{x=1}=(1,2,1)$, $v_{x=-2}=(-1,1,1)$

5) $x=1$, $x=4$, $x=-1$

$v_{x=1}=(1,-2,1)$, $v_{x=4}=(1,1,1)$, $v_{x=4}=(1,1,1)$

6) $x=-1$, $x=3$

$v_{x=-1}=(-1,2)$, $v_{x=3}=(1,2)$

7) $x=1+2i$

$v_{x=1+2i}=(1+i,2)$

8) $x=1$, $x=1+\sqrt{3}i$, $x=1-\sqrt{3}i$

$v_{x=1}=(1,1,1)$, $v_{x=1+\sqrt{3}i}=\left(1-\sqrt{3}i,1+\sqrt{3}i,-2\right)$, $v_{x=1-\sqrt{3}i}=(1+\sqrt{3}i,1-\sqrt{3}i,-2)$

1st Order Homogeneous with Constant Coefficients – Diagonalization

Solve the system:

1) $\underline{x}'(t)=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}\underline{x}(t)$

2) $\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}=\underbrace{\begin{pmatrix}-1&3&0\\3&-1&0\\-2&-2&6\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\ ;\ \underline{x}(0)=\begin{pmatrix}-1\\5\\3\end{pmatrix}$

3) $\begin{pmatrix}x'(t)\\y'(t)\\z'(t)\end{pmatrix}=\underbrace{\begin{pmatrix}4&-1&-1\\1&2&-1\\1&-1&2\end{pmatrix}}_{A}\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix}\ ;\ \underline{x}(0)=\begin{pmatrix}3\\2\\2\end{pmatrix}$. Prove that $z(t)=y(t)$.

Solve the system:

4) $\begin{aligned}x'&=x-y+4z\\y'&=3x+2y-z\\z'&=2x+y-z\end{aligned}$

5) $\underline{x}'(t)=\begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}\underline{x}(t)$

6) Given the system: $\underbrace{\begin{pmatrix}x'(t)\\y'(t)\end{pmatrix}}_{\underline{x}'(t)}=\underbrace{\begin{pmatrix}1&1\\4&1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x(t)\\y(t)\end{pmatrix}}_{\underline{x}(t)}\ ;\ \underline{x}(0)=\begin{pmatrix}1\\6\end{pmatrix}$.

Compute: $\lim\limits_{t\to\infty}\frac{y(t)}{x(t)}+\lim\limits_{t\to-\infty}\frac{y(t)}{x(t)}$

Solve the system:

7) $\begin{cases}y_1'+5y_1-2y_2'=0\\3y_2'-4y_1'-5y_2=0\end{cases}$

8) $\vec{x}'(t)=A\cdot\vec{x}(t)$ where $A=\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix}$

Final Answers:

1) $x_1(t) = -c_1 + c_3e^{2t}$, $x_2(t) = c_2e^t$, $x_3(t) = c_1 + c_3e^{2t}$

2) $\underline{x}(t) = e^{6t}\begin{pmatrix}0\\0\\1\end{pmatrix} + 2e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix} + 3e^{-4t}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

3) Prove.

4) $\underline{x}(t) = c_1e^{1t}\begin{pmatrix}-1\\4\\1\end{pmatrix} + c_2e^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix} + c_3e^{-2t}\begin{pmatrix}-1\\1\\1\end{pmatrix}$

5) $\underline{x}(t) = c_1e^{1t}\begin{pmatrix}1\\-2\\1\end{pmatrix} + c_2e^{4t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix}$

6) $\lim\limits_{t\to\infty}\frac{y(t)}{x(t)} + \lim\limits_{t\to-\infty}\frac{y(t)}{x(t)} = 0$

7) $\underline{x}(t) = c_1e^t\left[\cos 2t\begin{pmatrix}1\\2\end{pmatrix} - \sin 2t\begin{pmatrix}1\\0\end{pmatrix}\right] + c_2e^t\left[\cos 2t\begin{pmatrix}1\\0\end{pmatrix} + \sin 2t\begin{pmatrix}1\\2\end{pmatrix}\right]$

8) $\underline{x}(t) = c_1e^{1t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2e^t\left[\cos\sqrt{3}t\begin{pmatrix}1\\1\\-2\end{pmatrix} - \sin\sqrt{3}t\begin{pmatrix}-\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\right] + c_3e^t\left[\sin\sqrt{3}t\begin{pmatrix}1\\1\\-2\end{pmatrix} + \cos\sqrt{3}t\begin{pmatrix}-\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\right]$

1st Order Nonhomogeneous with Constant Coefficients - Variation of Parameters

Solve the following equations:

1) $x_1' = x_1 + x_2 + 2e^{-t}$
$x_2' = 4x_1 + x_2 + 4e^{-t}$

2) $x_1' = x_1 + x_2 + e^{at}$
$x_2' = 4x_1 + x_2 - 2e^{at}$

3) $\underline{x}'(t) = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 18t \\ 3 \\ 0 \end{pmatrix}$

4) $x' = x + y + 2z + e^t$
$y' = x + 2y + z$
$z' = 2x + y + z + e^t$

5) Convert the equation $y''' + y'' - 2y = t^2$ to a system of first order linear ODEs (no need to solve).

Final Answers:

1) $$\underline{x}(t) = \underbrace{c_1 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\underline{x}_h(t)} \underbrace{- \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\underline{x}_p(t)}$$

2) $$\underline{x}(t) = \begin{cases} c_1 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} te^{-t} \\ -2e^{-t} \end{pmatrix} & ;\ a = -1 \\ c_1 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{a+1}\begin{pmatrix} e^{at} \\ -2e^{at} \end{pmatrix} & ;\ a \neq -1 \end{cases}$$

3) $$\underline{x}(t) = c_1 e^{t}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + (3t+2)\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} - (3t+1)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + (-3t+1)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

4) $$\begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \left(\frac{1}{3}te^{t}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \left(-\frac{2}{9}e^{t}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

5) $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t^2 \end{pmatrix}$$

The Substitution Method:

Solve the system:

1) $\begin{cases} y''+2z'=e^{3x} & (1) \\ y'-z''+3z=x^2 & (2) \end{cases}$

2) $\begin{cases} y''+z'=e^{-2x} & (1) \\ y+z=\sin x & (2) \end{cases}$ $z(0)=y(0)=y'(0)=0$

3) $\begin{cases} x'=4x-2y-e^{t} \\ y'=6x-3y+e^{-t} \end{cases}$

4) $\begin{cases} x_1'=x_1+x_2+\sin 2t \\ x_2'=x_1+x_2+\cos 2t \end{cases}$

5) $\begin{cases} z''-3z'+2z+y'-y=0 \\ z'-2z+y'+y=0 \end{cases}$

Final Answers:

1) $y=\frac{1}{12}e^{3x}-\frac{2}{3}x^3-2c_2e^{x}+2c_3e^{-x}+kx+l$

2) $z=\frac{1}{2}+\frac{1}{6}e^{x}-\frac{1}{6}e^{-2x}-\frac{1}{2}\cos x+\frac{1}{2}\sin x$

3) $y=2c_1+\frac{3}{2}c_2e^{t}+6te^{t}-\frac{3}{2}e^{t}-\frac{5}{2}e^{-t}$

4) $x_1=c_1+c_2e^{2t}-\frac{1}{2}\cos 2t-\frac{1}{4}\sin 2t$, $x_2=-c_1+c_2e^{2t}+\frac{1}{4}\sin 2t$

5) $y=2c_1+\frac{1}{2}c_2e^{x}$